A THERMALLY EXCITED NON-LINEAR OSCILLATOR

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ABSTRACT

Instability in the form of growing oscillations (overstability) can occur in convectively unstable fluids which rotate, have magnetic fields, or are compressible, so long as thermal dissipation operates. To clarify the manner in which thermal dissipation causes instability, a model oscillator which exhibits overstability is constructed. The governing equations are derived and the linear stability is discussed.

The non-linear behavior of the oscillator is then investigated. The governing non-linear equation is third order in time, and it therefore is a simple representative example of the type of equation which describes non-linear stellar pulsations. The equation contains two parameters, and a great variety of solutions is found, depending on the values taken. One kind of solution shows relaxation oscillations with superposed variations, while, in a particular range of the governing parameters, the numerical solutions of the governing equation are aperiodic or irregular. A mathematical explanation of this irregularity is suggested, and the possibility that it might be relevant to irregular variability in stars is raised. The general conclusion is suggested that a great variety of oscillatory phenomena, analogous to several of those observed in variable stars, can be generated from a single instability mechanism, provided the essential non-linearities are retained and the law of dissipation is appropriately chosen.

I. INTRODUCTION

One of the most striking features of linear convection theory is the possibility that in a rotating system (or in one with a prevalent magnetic field) instability can occur in the form of growing oscillations. This behavior was predicted by Chandrasekhar (1953, 1961) who, following Eddington's characterization of pulsational instability in stars, called it "overstability."

Like pulsational instability, convective overstability does not occur if the motion is adiabatic. This can be seen directly by examining Chandrasekhar's result in the limit of vanishing thermal conductivity. Viscosity, on the other hand, does not play a vital role (though some authors have suggested that it does) and an inviscid fluid can be overstable so long as conduction (either thermal or radiative) acts.

The role of thermal dissipation in producing convective overstability is well illustrated by the case of a convectively unstable, rotating layer of fluid. When the fluid is both inviscid and non-conducting, Cowling (1951) has shown that two kinds of modes can occur. For motions with a horizontal wavenumber greater than a certain critical value (which depends on the angular velocity) the motion is convective and develops exponentially in time. On the other hand, motions with wavenumbers below the critical value oscillate neutrally. When thermal conduction is introduced, it damps the convective motions, but it also leads to an amplification of the oscillatory motions. A physical explanation of this destabilization of the oscillatory modes has been suggested by Cowling (1957). He pointed out that the heat exchange leads to asymmetries in the motion such that an oscillating particle returns to its equilibrium position faster than it leaves. The operation of this mechanism in each cycle causes continual amplification of the oscillations.

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We believe that the mechanism discussed by Cowling will operate whenever a restoring force introduces oscillatory modes in a convectively unstable fluid. We can anticipate, for example, that in a compressible medium, the oscillatory (acoustic) modes may also be overstable if thermal dissipation acts. Indeed, this suggestion has now been verified by detailed calculation for a polytropic atmosphere (Spiegel 1963; Kato 1965). This means that, in a stellar convection zone, sound waves may be generated directly by instability rather than secondarily by turbulence, and it opens up the possibility for a new mechanism of exciting high-frequency waves which can deliver mechanical energy to the corona.

Now the acoustic and convective modes in a plane-parallel atmosphere are just the p- and g-modes of pulsation theory, in the limit where the curvature is neglected. Thus, by analogy, we can anticipate that in a star with a convectively unstable zone, non-radial instability in the form of growing p-modes will occur because of the mechanism just mentioned.

Thus convective overstability seems to be a far-reaching kind of mechanism, and it is also an important example of a self-excited oscillation which relies on thermal dissipation for its excitation. A study of its non-linear aspects may serve to clarify the general problem of such self-excited oscillations. Accordingly, we have attempted to fix attention on one form of overstable oscillator, and to explore it as fully as we can. To do this we have devised a relatively simple model oscillator, which exemplifies the physical processes of convective overstability. We shall describe this oscillator in § II and devote § III of this paper to the treatment of its linear-stability theory. The discussion of the non-linear behavior of the oscillator then occupies the remainder of the paper. One of the chief purposes of the discussion is to illustrate the remarkable variety that can be exhibited by the behavior of non-linear dissipative oscillators.

II. AN OVERSTABLE OSCILLATOR

In order to clarify the role of thermal dissipation in convective overstability and to develop a specific model which may exemplify the non-linear aspects of overstability, we shall consider the oscillations of a small, compressible mass element attached to a fixed point by an elastic spring. This oscillator is immersed in a horizontally stratified fluid, and we shall introduce irreversible effects by allowing the element to exchange heat with the ambient fluid. For simplicity, let us take the mass element to consist of some of the ambient fluid confined by a membrane which transmits pressure fluctuations. We neglect the forces on the element due to the motion set up in the ambient fluid and assume that the only force exerted by the fluid on the element is a buoyancy force.

The equation of motion of the oscillating mass is then

$$M\frac{d^2z}{dt^2} = -g[M-\rho_0(z)V(t)]+Mr(z)\,, \tag{1}$$

where ρ_0 is the density of the ambient fluid, z is the displacement of the mass element, M is the (constant) mass of the element, V is its volume, and Mr is the restoring force of the spring. Since the mass of the element is constant,

$$M = \rho(t)V(t) = \rho_0(0)V_0\,, \tag{2}$$

where ρ is the density of the oscillator, and V_0 is its volume when its density is $\rho_0(0)$.

We will always assume that the displacements are much smaller than the scale height of density variations in the ambient fluid so that the density variations of the element will also be small. This is equivalent to the Boussinesq approximation of convection theory (see Chandrasekhar 1961). Thus, for example, we can write,

$$V(t) = V_0 - \frac{M}{\rho_0^2(0)}[\rho(t)-\rho_0(0)]\,. \tag{3}$$

With this approximation, the equation of motion becomes,

$$\rho_0(0)\frac{d^2z}{dt^2}=g[\rho_0(z)-\rho_0(0)]-g\frac{\rho_0(z)}{\rho_0(0)}[\rho(t)-\rho_0(0)]+\rho_0(0)r(z). \tag{4}$$

Equation (3) results from the application of the Boussinesq approximation to the mass element. If we now apply this approximation to the ambient fluid we can replace $\rho_0(z)$ by $\rho_0(0)$ in the middle term of equation (4); this leaves

$$\rho_0(0)\frac{d^2z}{dt^2}=g[\rho_0(z)-\rho_0(0)]-g[\rho(t)-\rho_0(0)]+\rho_0(0)r(z). \tag{5}$$

We may also apply the Boussinesq equation of state to the problem. This is

$$\rho_0(z)=\rho_0(0)\{1-\alpha[T_0(z)-T_*]\}, \tag{6}$$

for the ambient fluid, and

$$\rho(t)=\rho_0(0)\{1-\alpha[T(t)-T_*]\}, \tag{7}$$

for the element, where T_0 is the ambient temperature and T_* is the temperature at which ρ and ρ_0 equal $\rho_0(0)$. The equation of motion then becomes

$$\frac{d^2z}{dt^2}=g\alpha\theta+r(z), \tag{8}$$

where

$$\theta=T(t)-T_0(z) \tag{9}$$

is the temperature difference between the element and its immediate surroundings.

Equation (8) describes the dynamics of the oscillator, and if we now specify its thermal interaction with the ambient fluid we shall have a determinate system. To do this we shall assume that the membrane confining the fluid element is transparent, so that the element can interact radiatively with the ambient fluid. Let us also assume that the fluid element is so small that it is optically thin. In that case, Newton's law of cooling applies applies (Spiegel 1957), and we can write

$$\frac{dT(t)}{dt}=-q\theta, \tag{10}$$

where q^{-1} is a characteristic time for radiative damping.

We can now differentiate equation (8) with respect to time and introduce equation (10). We obtain

$$\frac{d^3z}{dt^3}-\left[\frac{dr}{dz}+g\alpha\beta(z)\right]\frac{dz}{dt}+g\alpha q\theta=0, \tag{11}$$

where

$$\beta(z)=-\frac{dT_0(z)}{dz}. \tag{12}$$

Then we can eliminate θ between equations (11) and (8) to obtain

$$\frac{d^3z}{dt^3}+q\frac{d^2z}{dt^2}-\left(g\alpha\beta+\frac{dr}{dz}\right)\frac{dz}{dt}-qr(z)=0. \tag{13}$$

Thus, if we now specify the restoring force, r, and the negative temperature gradient, β, as functions of z, the equation of motion is completely determinate, and the study of the overstable oscillations can proceed from equation (13).

III. STABILITY THEORY

We now wish to consider the initial behavior of the oscillator when it is displaced from equilibrium by a small amount. To do this, we shall treat z as a small quantity and neglect terms of second order. We have

$$r(z) = -\lambda^2 z + O(z^2)\,, \tag{14}$$

and

$$\beta = \beta_0 + O(z)\,, \tag{15}$$

where λ and β_0 are real constants. To first order in small quantities, equation (13) then becomes

$$\frac{d^3 z}{dt^3} + q\,\frac{d^2 z}{dt^2} + \left(\lambda^2 - g\alpha\beta_0\right)\frac{dz}{dt} + q\lambda^2 z = 0\,. \tag{16}$$

Equation (16) admits solutions of the form exp (σt) where σ then satisfies the cubic equation

$$\sigma^3 + q\sigma^2 + (\lambda^2 - g\alpha\beta_0)\sigma + q\lambda^2 = 0\,. \tag{17}$$

This equation is identical to the cubic equation which arises in the theory of convection in a rotating system, and the spring constant λ plays an analogous role to that of the angular velocity. The discussion of the roots of the cubic equation is quite similar in the two cases, and the rotating case has already been discussed in detail (Veronis 1959; Chandrasekhar 1961; Danielson 1961; Weiss 1964).

Consider first the adiabatic case where $q = 0$. Equation (17) then has the three solutions

$$\sigma_0 = 0,\ \pm\,(g\alpha\beta_0 - \lambda^2)^{1/2}\,. \tag{18}$$

If $\beta_0 < 0$, the fluid is convectively stable and oscillatory motions always occur. For convective instability, $\beta_0 > 0$, and oscillations occur only for sufficiently strong springs, just as in a rotating system where oscillations occur only for sufficiently large rotations. For weak springs, the solutions are exponential in time, in analogy with the convective modes.

When q is small we can easily show that the three roots of equation (17) are

$$\sigma = -\frac{q\lambda^2}{(\lambda^2 - g\alpha\beta_0)} + O(q^2)\,, \tag{19}$$

and

$$\sigma = \pm\,(g\alpha\beta_0 - \lambda^2)^{1/2} + \frac{g\alpha\beta_0 q}{2(\lambda^2 - g\alpha\beta_0)} + O(q^2)\,. \tag{20}$$

Let us first consider the two roots given in equation (20). If $\beta_0 < 0$ the thermal dissipation is always stabilizing. But if $\beta_0 > 0$ and $\lambda^2 > g\alpha\beta_0$, so that the adiabatic system would oscillate rather than convect, the thermal dissipation is destabilizing. The oscillations grow slowly in amplitude with growth rate $O(q)$. This establishes the overstable character of the oscillator, and shows how it is a direct result of thermal dissipation. A physical explanation, which is an elaboration of that given by Cowling (1957) for the hydromagnetic case, is as follows for the case $\lambda^2 > g\alpha\beta_0$.

Imagine the mass to be at the origin and moving upward into the cooler, denser fluid. If there were no thermal dissipation, the element would retain its initial density. Thus, when it returned to the origin under the net restoring force, it would be in the same state as when it left, but now it would be moving downward. The element would then execute symmetric oscillations. However, for finite thermal dissipation, the element cools as it rises. As it passes upward through the origin it has a density deficiency with respect to its surroundings since it has just come from a warmer region. However, it now passes upward into a cooler region and its temperature drops. Because of the finite relaxation time of temperature adjustments associated with equation (10), the element will return to the origin with a higher density, hence a smaller density deficiency, than when it last left the origin. The upward buoyancy force on its upward passage through the origin is therefore greater than on its downward passage. A similar difference in buoyancy occurs between the upward and downward passage through each point on the upper half-cycle. This means that the element will return to the origin moving faster than when it left on each swing upward from the origin. On the downward swings, the same effect operates, but with opposite signs. In this way the amplitude of the oscillations will build up exponentially.

If $\lambda^2 < g\alpha\beta_0$, oscillations are impossible. In this case, the mass element rises or sinks without being stopped (in the linear approximation), but the radiation leads to a reduction of the velocity of rise. This corresponds to the reduction of convective growth rate normally associated with thermal dissipation.

The third root, that given in equation (19), represents a slow drift downward in the strong-spring case, $\lambda^2 > g\alpha\beta_0$, and a slow drift upward in the weak-spring case. This may also be simply explained. In the adiabatic case, the element will be in equilibrium at any point where its buoyancy is canceled by the restoring force of the spring. Suppose the element is lighter than the ambient fluid at $z = 0$. Recalling that, for $\beta_0 > 0$, the density of the ambient fluid increases with height, we see that the element will be in equilibrium below $z = 0$ if the spring is weak and above $z = 0$ if the spring is strong. A small thermal dissipation will gradually destroy the element's buoyancy and the spring will slowly pull the element toward $z = 0$, giving the motions predicted by equation (19). This type of motion is important in the non-linear case treated in the subsequent sections.

The above physical argument applies for arbitrary values of q, but a more general discussion of equation (17) is required to provide the roots of the cubic. This is facilitated by putting equation (17) in non-dimensional form. Let the cooling time, q^{-1}, be the unit of time, and let us introduce the notation

$$R = \frac{g\alpha\beta_0}{q^2} \tag{21}$$

and

$$T = \left(\frac{\lambda}{q}\right)^2 . \tag{22}$$

Equation (17) then becomes

$$\sigma^3 + \sigma^2 + (T - R)\sigma + T = 0 , \tag{23}$$

where, by comparison with the theory of inviscid convection in a rotating fluid, we see that T is analogous to the Prandtl number times the Taylor number, while R is analogous to the Prandtl number times the Rayleigh number. Alternatively, we can say that T is the square of the ratio of the radiative relaxation time to the free period of the spring, while R is the square of the ratio of relaxation time to free-fall time.

Let

$$\Delta(T,R) = 4R^3 + R^2(1 - 12T) + 12T^2 - 20T - 4T^3 - 8T^2 - 4T . \tag{24}$$

Straightforward analysis shows that if $\Delta(T,R) > 0$, then equation (17) has two positive real roots and one negative real root, while if $\Delta(T,R) < 0$, equation (17) has one negative real root and a pair of complex conjugate roots with positive real parts (see Fig. 1). The curve, $\Delta(T,R) = 0$, divides the positive quadrant of the (T,R)-plane into two parts, the upper part corresponding to convective-like motions, the lower to unstable oscillations.

For large T and R (i.e., small q) the dividing curve is asymptotic to the straight line, $T = R$, since it is this curve which gives the transition from oscillatory to convective modes in the adiabatic case. For finite T and R, however, the curve $\triangle(T,R) = 0$ lies above this straight line, indicating that thermal dissipation permits oscillatory modes for smaller spring constants than are required in the nearly adiabatic case. This is due to the reduction of buoyancy force caused by thermal dissipation.

The foregoing discussion shows both mathematically and physically how thermal dissipation (in this case radiative transfer) leads to a destabilization of oscillatory motion in a convectively unstable fluid. The analogy to acoustic modes (p-modes) and to inertial oscillations (for the rotating case), made in the Introduction, is now easier to justify.

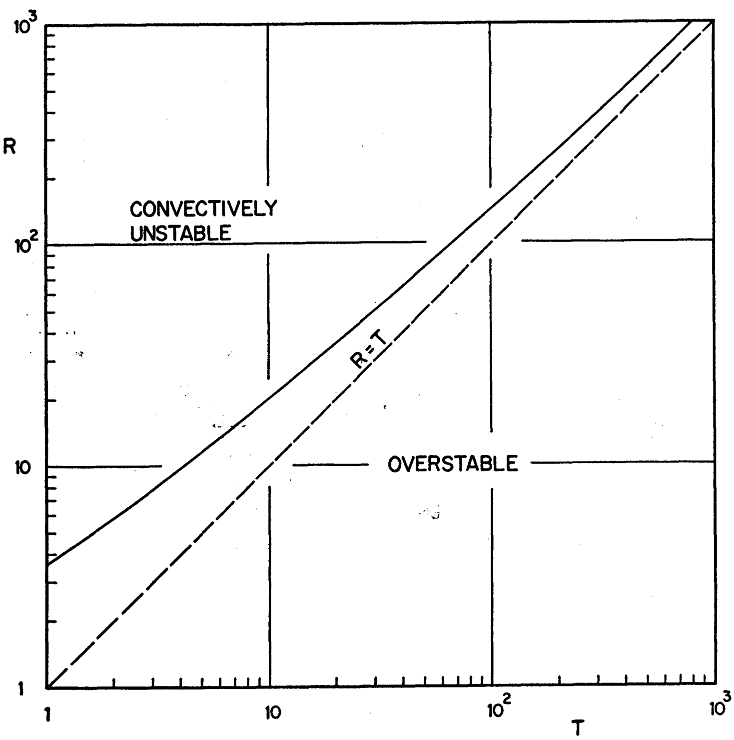

FIG. 1.—The regions of instability in the R–T plane. The solid curve is $\Delta(T,R) = 0$; the broken line is $R = T$.

The oscillating mass of the model corresponds to a typical particle moving in an oscillatory mode. In the case of the p-mode, the pressure force plays the role of the restoring force of the spring.

IV. NON-LINEAR TEMPERATURE PROFILE

Having established a simple example of a self-exciting oscillator, which depends on thermal dissipation in an essential way, we can proceed to inquire into its non-linear behavior. The effect of non-linearity which is of greatest interest is the limitation of the motion to finite amplitude. Now it can be seen from the physical discussion that a hard spring will not by itself prevent the oscillations from growing indefinitely, since the precise shape of the oscillations will not alter the direction of the energy transfer. Thus we shall keep the restoring force linear. However, if the unstable fluid is bounded on both sides by stably stratified fluid, when the element penetrates these stable regions it will have to give up energy. In this way the oscillations will be limited in amplitude. The situation is similar to that in stars where convectively unstable regions are always bounded by stable regions.

The simplest example of a non-linear temperature profile of a suitable shape is given by

$$\beta(z) = \beta_0[1 - (z/L)^2] , \tag{25}$$

where L is an arbitrary length. For $|z| < L$ the fluid is then unstably stratified, while for $|z| > L$ it is stable. Henceforth, it will be convenient to let L be the unit of length, which is equivalent to putting $L = 1$. The spring force is still given by

$$r(z) = -\lambda^2 z . \tag{26}$$

Equation (13) becomes

$$\frac{d^3z}{dt^3} + q\frac{d^2z}{dt^2} - [(g\alpha\beta_0 - \lambda^2) - g\alpha\beta_0 z^2]\frac{dz}{dt} + q\lambda^2 z = 0 , \tag{27}$$

which will be the basis of the remainder of this discussion. In § VII we will give numerical solutions of this equation, but the discussion of these solutions is facilitated if we first treat some limiting cases. This we do in the next two sections.

V. ADIABATIC MOTION

In this section we shall neglect thermal dissipation completely. We then have a non-dissipative, non-linear oscillator which is not unlike simple oscillators which have been studied previously, and it is possible to show that every motion of the oscillator is periodic with finite amplitude.

If $q = 0$, the equation of motion becomes

$$\frac{d^3z}{dt^3} + [\lambda^2 - g\alpha\beta_0(1 - z^2)]\frac{dz}{dt} = 0 . \tag{28}$$

It is convenient here to let $(g\alpha\beta_0)^{-1/2}$ be the unit of time. Equation (28) then becomes

$$\frac{d^3z}{dt^2} + (\Lambda - 1 + z^2)\frac{dz}{dt} = 0 , \tag{29}$$

where

$$\Lambda = \frac{\lambda^2}{g\alpha\beta_0} = \frac{T}{R} \tag{30}$$

is a non-dimensional measure of the spring strength.

Equation (29) can be integrated at once, and we find

$$\frac{d^2z}{dt^2} + (\Lambda - 1)z + \tfrac{1}{3}z^3 = b , \tag{31}$$

where b is a constant of integration.

Clearly, b is a measure of the force on the element when it is at the origin, and b is therefore related to the density of the element. Since thermal exchange is neglected, this density is a constant of the motion. (In the case where $b = 0$, eq. [31] reduces to an equation considered by Stoker [1950] in connection with the buckling of an elastic column.)

In this adiabatic limit, the oscillator has positions of equilibrium which are given by the roots of the cubic equation,

$$(\Lambda - 1)z + \tfrac{1}{3}z^3 = b . \tag{32}$$

Analysis of the cubic shows that for $\Lambda > 1$ it has only one root, whatever b is, while for $\Lambda < 1$ it has one real root only when $|b|$ is greater than

$$b_c = \tfrac{2}{3}(1 - \Lambda)^{3/2} . \tag{33}$$

For $\Lambda < 1$ and $|b| < b_c$ there are three positions of equilibrium, one lying in the interval $|z| < (1 - \Lambda)^{1/2}$, and one on either side of this interval.

If z_0 is any of these equilibrium points, the character of its equilibrium may be investigated by studying solutions of the form $z = z_0 + y$ for small y. Equation (31) then yields

$$\frac{d^2 y}{dt^2} + (\Lambda - 1 + z_0^2)\, y = 0\,. \tag{34}$$

We see from this that an equilibrium point in the region $|z| < (1 - \Lambda)^{1/2}$ is unstable while equilibrium points outside this region are stable. We can then infer that in the phase plane $(z, dz/dt)$ the case $\Lambda < \Lambda_c$ will have two centers with a saddle point between them, while for $\Lambda > \Lambda_c$ we have simply one center. (For $\Lambda = \Lambda_c$ eq. [32] has coincident roots, and we have higher-order singularities at the position of equilibrium. In the analogous problem of the buckling of a column, this case corresponds to the critical condition for buckling.)

Since equation (31) represents a non-dissipative motion, we can expect energy conservation. The energy equation is obtained by multiplying equation (31) by dz/dt and integrating once. We then obtain

$$\left(\frac{dz}{dt}\right)^2 + (\Lambda - 1)\, z^2 + \frac{z^4}{6} - 2bz = 2E. \tag{35}$$

Figure 2 illustrates these solution-curves in the two possible cases of one and three positions of equilibrium. All the solution-curves are bounded and closed. Thus, every possible adiabatic motion of the oscillator, with a single exception, is periodic and of finite amplitude. The exception corresponds to the figure-eight curve which enters the saddle point, and represents an asymptotic approach to unstable equilibrium.

VI. SMALL CONDUCTIVITY AND WEAK SPRING

If we return to the general, non-adiabatic case, we must now base our considerations on equation (27). This equation has three coefficients in it, each of which is a characteristic time. These are: (*a*) q^{-1}, the radiative relaxation time, (*b*) λ^{-1}, the time constant of the spring, and (*c*) $(g\alpha\beta_0)^{-1/2}$, the free-fall time, reduced by buoyancy. The general case in which these times have arbitrary values is to be treated in the next section by numerical methods. But the case where the spring is weak and radiative effects are small can be treated approximately, and is both interesting in itself and quite useful in understanding the more general numerical results.

Let us therefore consider what happens when the characteristic times of radiation (*a*) and of the spring (*b*) are very much longer than the free-fall time (*c*). In terms of the parameters we have used previously, this means that $\Lambda \ll 1$, $R \gg 1$. For definiteness, let us suppose that the element is initially at rest at its upper position of stable equilibrium, with $z_0 > 0$. If at this initial instant a small radiative term is turned on, what will happen?

Since the element was in stable equilibrium initially, its initial density must be less than that of its surroundings so that the buoyancy is upward and holds the element in place against the downward tension of the spring. On account of the radiation, the element will tend toward thermal equilibrium with its surroundings, its density will rise, and it will slowly lose buoyancy. We can picture the element as being pulled slowly down by the spring as its buoyancy diminishes. At each point it will seek thermal equilibrium and continue to lose buoyancy, but, as the spring is weak, the sinking is quite slow, and the element will sink through a sequence of states of static equilibrium. The accelerations in equation (27) can thus be thought of as small perturbations acting on the quasi-static solution.

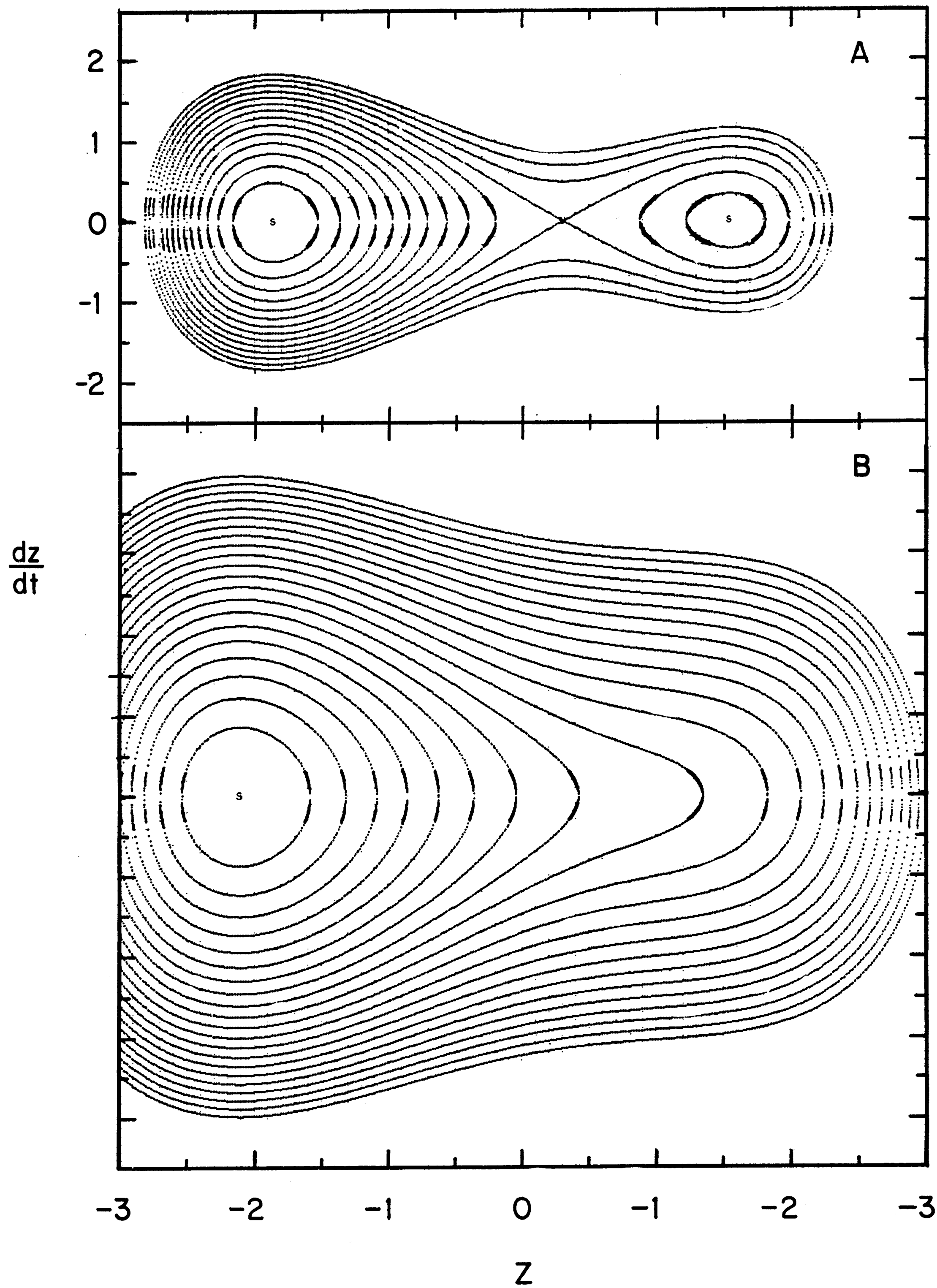

FIG. 2.—Solution curves in the phase plane computed from equation (35) for the case $\Lambda = 0$ and $b = -0.3$ (*upper curves*) and -1.0 (*lower*).

As the element continues to sink, it finally passes the point $z = (1 - \Lambda)^{1/2}$; but we saw that equilibria in the region $|z| < (1 - \Lambda)^{1/2}$ are unstable. Thus, since we have taken $\Lambda \ll 1$, we see that, when the element comes into the region $|z| \lesssim 1$, where the fluid has an unstable density distribution, its static equilibrium state becomes unstable. Once the element is in the unstable region it will accelerate downward and pass through the unstable region quickly in a time $(g\alpha\beta_0)^{-1/2}$. The slowness of the thermal adjustment will completely prevent the element from coming to equilibrium in these new surroundings, and thus it retains the density and temperature it had at $z \simeq 1$.

When the element crosses back into stable fluid at $z = -1$ it will therefore be heavier than its surroundings and will continue to be impelled downward by the buoyancy force. Now, the ambient fluid has the same density at $z = -2$ as at $z = +1$. (This follows from eq. [25] which implies that T_0, hence ρ_0, behaves like $z - \frac{1}{3}z^3$.) Therefore, the element can be in equilibrium at $z = -2$ with its present density and will execute decaying oscillations about $z = -2$ with decay time $O(q^{-1})$. In fact, the adiabatic solutions show that it will initially overshoot to $z = -3$, so that the initial amplitude is large.

The element will now begin to creep upward through a sequence of equilibrium states until it reaches $z = -1$, the lower boundary of the unstable region. It then moves quickly across the unstable layer, executes damped oscillations about $z = +2$, and starts to creep down again. Thus it goes into a limit cycle of the relaxation type. The period of the motion will be very nearly the time spent in the creeping motion, and we can estimate this by neglecting the acceleration term in the dynamical equation. If we do this, equation (8) becomes

$$g\alpha\theta - \lambda^2 z = 0\,, \tag{36}$$

where equation (26) has been used. If we combine this with equations (9) and (10), we get

$$[\,g\alpha\beta_0(1-z^2) - \lambda^2\,]\,\frac{dz}{dt} = q\lambda^2 z\,. \tag{37}$$

We may now use this result to estimate the period, P, of the cycle, which is just twice the time for the element to creep from $z = 2$ to $z = 1$; that is,

$$P = 2\int_2^1 \frac{dz}{dz/dt} \cong \frac{2g\alpha\beta_0}{q\lambda^2}\int_2^1 \frac{1-z^2}{z}\,dz = (3 - 2\ln 2)\,\frac{R}{Tq} \cong 1.61\,\frac{R}{qT}\,. \tag{38}$$

The condition that the quasi-static approximation is valid is that the period P be much larger than the cooling time q^{-1}. Thus we must have $R \gg T$.

If we now measure times in units of cooling times, we can summarize the results as follows. We have a basic relaxation oscillation of period $O(R/T) = O(1/\Lambda)$. Superimposed on the basic relaxation oscillation are transients of period $O(R^{-1/2})$, which are excited twice during each cycle and which decay with a period $O(1)$. This is confirmed by numerical integrations of equation (27). Figure 3 shows the solution for $R = 100$, with T ranging from 5 to 25. The periods for the lower values of T (5 and 10) are in reasonable agreement with those found above.

VII. THE GENERAL CASE

We turn now to the examination of the general solutions of equation (27). If we measure time in units of the cooling time, this equation becomes

$$\frac{d^3z}{dt^3} + \frac{d^2z}{dt^2} + (T - R + Rz^2)\,\frac{dz}{dt} + Tz = 0\,, \tag{39}$$

where R and T are defined in equations (21) and (22).

In the previous section we saw that, when R is very large and T is of order unity, equation (27) leads to a limit cycle which can be discussed theoretically. The assumption $R \gg T$ insured that the transient oscillations, which are excited every time the element crosses the unstable region, decay in a time short compared to the time of creeping though a sequence of equilibrium configurations in the stable region. We saw also that the time spent in the stable regions is $O(R/T)$. Thus, as T increases, the element spends decreasing amounts of time in the stable regions so that the transients occupy an increasing fraction of the cycle. As T is increased, then, a situation is reached in which the transients persist for the whole time the element is in the stable region.

This general behavior is confirmed by numerical integrations of equation (39). For the case R fixed and equal to 100 we can see this behavior as T is increased from zero. In

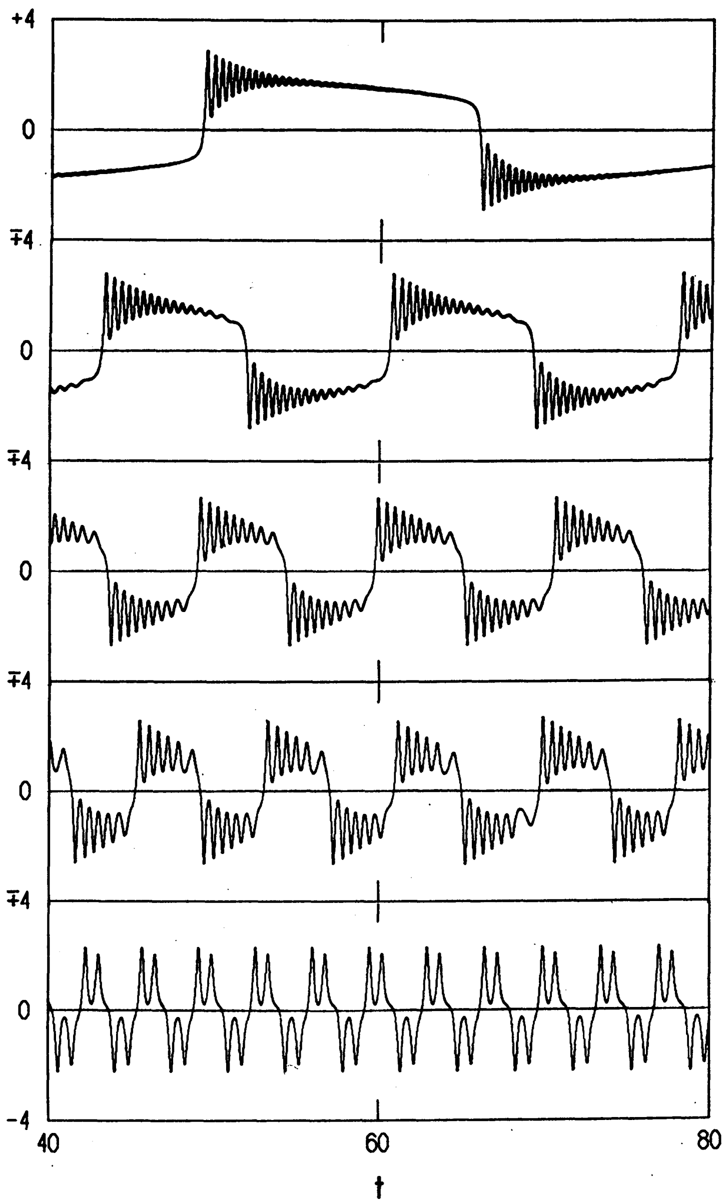

FIG. 3.—Amplitude versus time for weak springs. In each case $R = 100$. In descending order the solutions are for the values $T = 5, 10, 15, 20, 25$.

Figure 3 we show numerical solutions of equation (39) for $T = 5, 10, 15, 20$, and 25, for $R = 100$. In each case the equation was integrated until the form of solution did not change with time. These solutions show the features discussed in the preceding section: relaxation oscillations with shorter-period transient oscillations occurring periodically twice in each cycle. The superposed transient oscillations are analogous to the phenomenon of squegging, which has been found experimentally in work on electron tube oscillators. As T is increased, for fixed R, the number of squegging oscillations per limit cycle decreases, until at $T = 25$ the limit cycle has only two peaks.

One might anticipate that this trend would continue, but when T is increased to 26 a surprising change in the nature of the solutions takes place. Figure 4 shows that the motion does not settle down into a limit cycle but remains aperiodic. (In some cases we

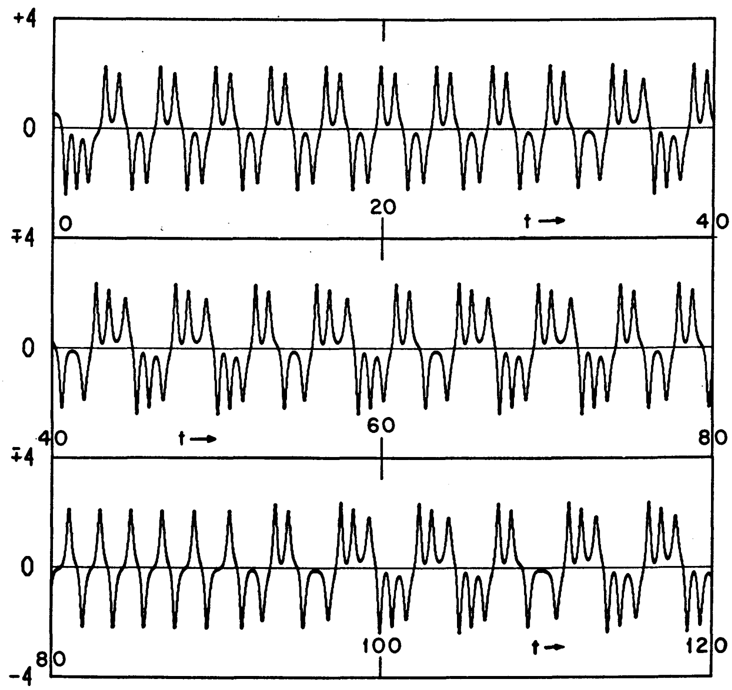

FIG. 4.—Irregularity of the solution for the case $R = 100$, $T = 26$. The three sections represent a continuous run over 120 cooling times.

integrated for as many as 1000 cooling times.) This aperiodicity persists as T is increased until $T = 40$, when a periodic asymmetric solution is found. The asymmetry gradually decreases as T increases further, and for $T \geq 50$ a single-peaked symmetric limit cycle occurs. For larger T the motion is controlled more and more by the spring, and eventually sinusoidal solutions are found. Figure 5 shows results for $T = 30, 35, 39$, and 40. It is noticeable that toward the larger values of T in the range of aperiodicity, there is a tendency toward regularity. The solution for $T = 39$ resembles that for $T = 40$, but appears to be randomly distorted; at $T = 26$ the lack of periodicity is more marked and the solution looks as if three different kinds of solutions are competing. In Figure 6 we show the case $T = 45$, for $R = 100$; a simple periodic behavior is exhibited for this case as for all higher values of T.

We found similar behavior for other large values of R. In Figure 7 we show the approximate region of the (R,T)-plane in which aperiodicity occurs. Above this region we always found solutions of the relaxation type whereas below the region we found single-peaked solutions which were asymmetric near the lower boundary of the region and symmetric elsewhere. Below a critical R, which we estimate as 11, there was no aperiodicity.

The aperiodicity of the solutions is interesting, and we would like to speculate on the reason for its occurrence. It cannot be ruled out without further analysis that higher-order periodicities exist but have not been recognized. Indeed, we have found some examples in which very complicated periodic solutions exist, but which might well be classified as aperiodic if viewed over a slightly shorter interval of time. Clearly, we need to formulate some general criterion for aperiodicity, but for the present qualitative discussion, in the absence of a recognizable periodicity, we shall consider a solution aperiodic.

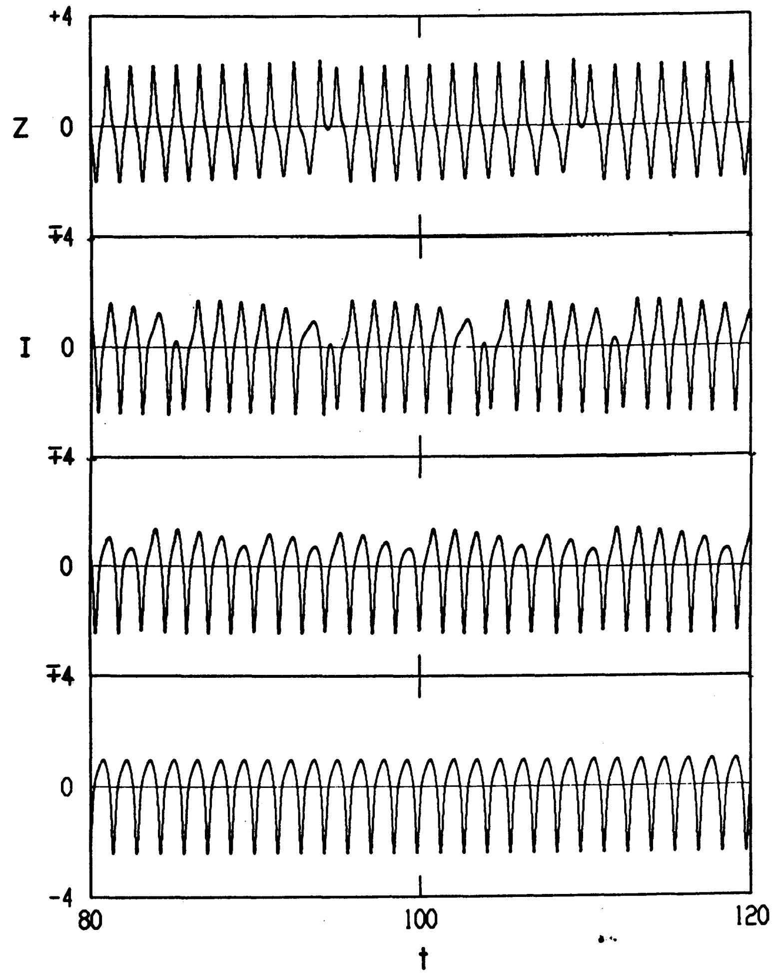

FIG. 5.—Solutions in the aperiodic range. $R = 100$ for all cases and, in descending order, $T = 30$, 35, 39, 40.

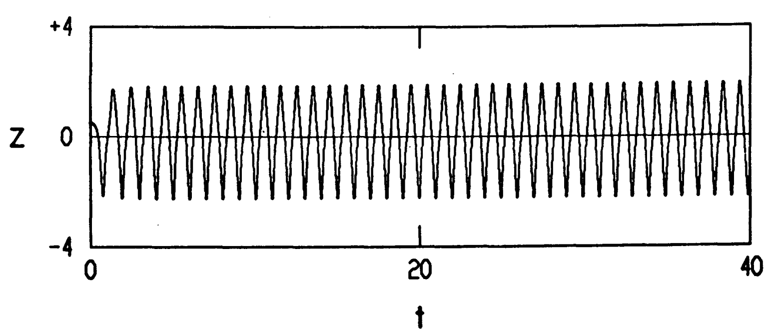

FIG. 6.—The periodic behavior for stronger springs. This case is $R = 100$, $T = 45$

Our discussion of the aperiodic oscillations starts from the observation that equation (40) defines the possible trajectories of the system in a three-dimensional phase space (z, dz/dt, d^2z/dt^2). We also note that, according to equation (40), the origin in this space is a point of unstable equilibrium. Our belief is that equation (40) admits solutions which are periodic even when R and T are in the ranges where the numerical solutions are aperiodic. Suppose, however, that there are several periodic solutions for each pair (R,T), and that the various trajectories pass near to the origin in phase space. In that case, if the representative point of the system in phase space moves along one of these trajectories, when it is in the neighborhood of the origin, a small perturbation may shift it into another. However, trajectories in the phase space, which are neighbors near the unstable point, may subsequently diverge. If this is the case here, then a mechanism to explain the aperiodicity is provided. In the numerical integrations, as in nature, small perturbations are always present (even though, here, double-precision arithmetic was employed and the errors in z, dz/dt, and d^2z/dt^2 were less than 10^{-7}), and these can effect large changes in the solution by moving the representative point to different trajectories.

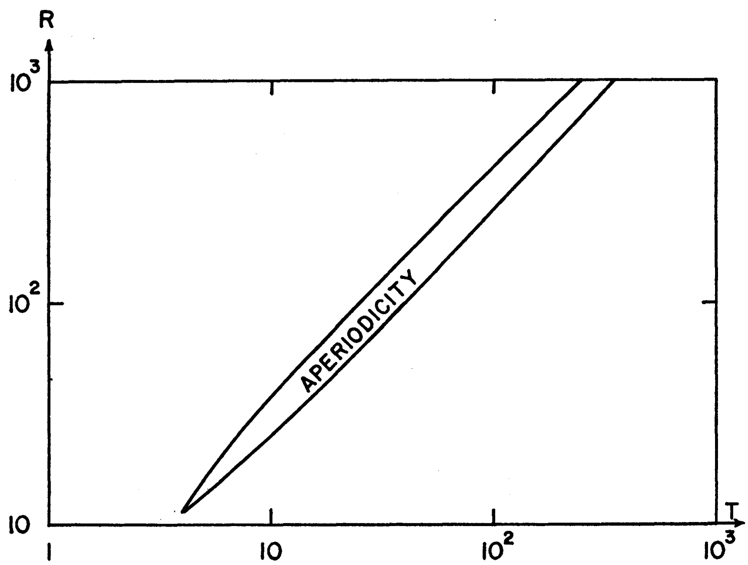

FIG. 7.—The region of aperiodicity in the R–T plane

We can assemble the following support for our suggestion that the aperiodicity arises because there are several periodic solutions of equation (40) which pass close to the origin:

a) In the case $R = 20$, aperiodicity set in when $T = 6$. Figure 8 shows z, dz/dt, and d^2z/dt^2 for the case $R = 20$, $T = 5.8$. The "stillstand" on the z-curve is characteristic of a close approach to the origin and is a sign of incipient instability.

b) In order to swamp the small perturbations arising from numerical errors, we introduced on the right-hand side of equation (40) a random term whose magnitude could be controlled. The random term took the form of impulses introduced with random frequency and sign. The results in the case $R = 20$, $T = 6$ are shown in Figure 9. It will be observed that the solutions, for no random term (*a*), and for a random input of magnitude 10^{-2} (*b*), agree perfectly until the "stillstand" on the unperturbed curve is reached. The solutions diverge at this point and are completely different subsequently.

c) Dr. N. H. Baker has developed a numerical method to find periodic solutions of non-linear differential equations. Using this method he has shown (private communication) that equation (40) possesses a symmetric and several asymmetric solutions in some

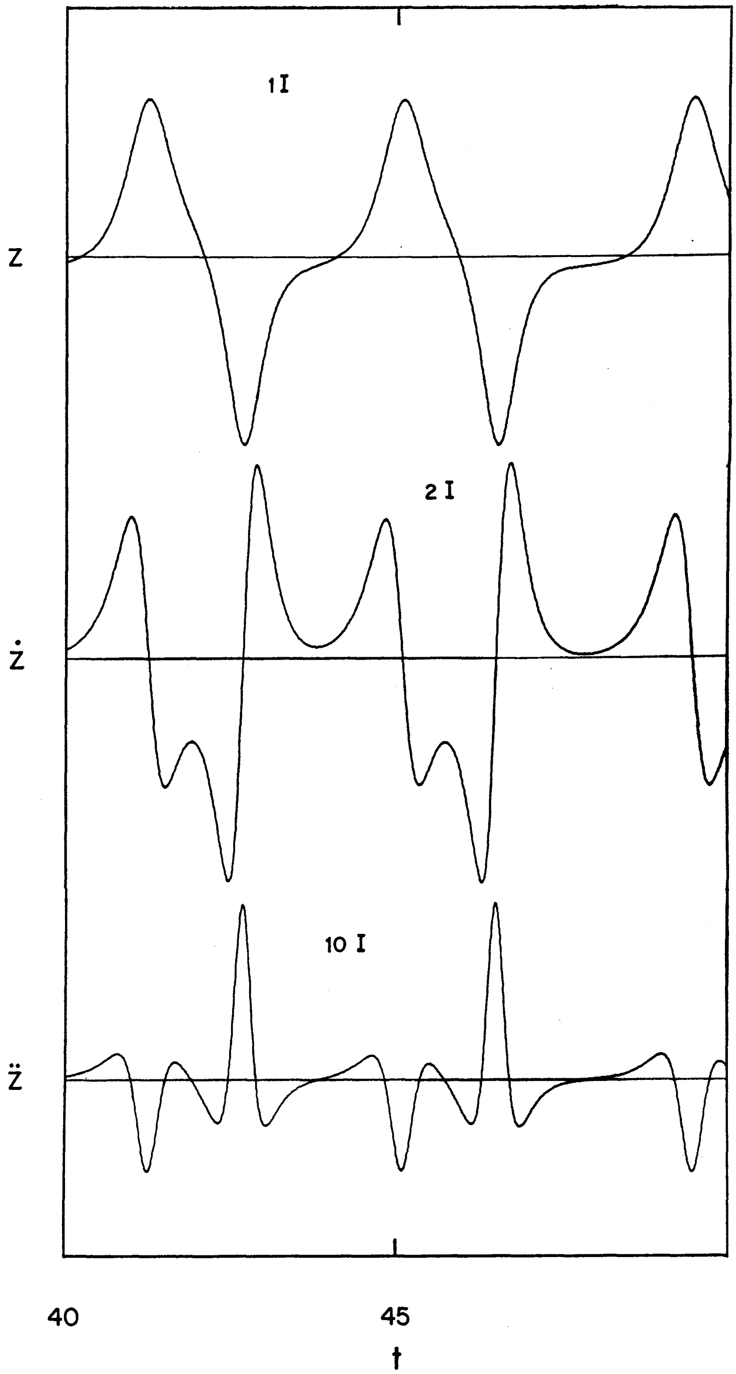

FIG. 8.—The solution for $R = 20$ and $T = 6$ showing a case just above the aperiodic region in the R–T plane. The scale for each curve is indicated by a line segment.

ranges of R/T. In the region of aperiodicity the solutions have pronounced "stillstands." We can think of the marked aperiodicity at the upper boundary of the unstable region of Figure 7 as a vacillation between several possible unstable solutions.

Baker's method shows that there is a symmetric solution in the region of periodic asymmetric solutions. The symmetric solution never occurred in our time integrations, though sometimes the initial stages of the motion were approximately symmetric.

Work on the aperiodicity is continuing, and we hope to report our results later. In particular it remains to explain why the aperiodicity is restricted to a well-defined range of values of T for each R.

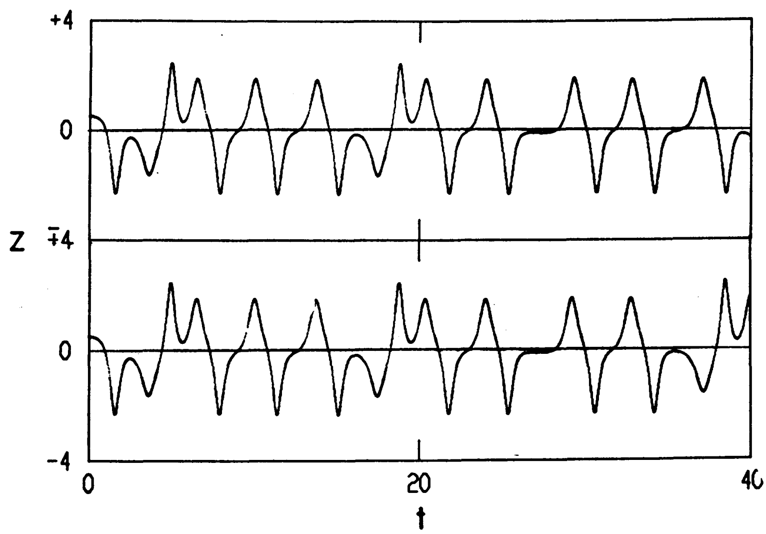

FIG. 9.—The amplitude of oscillation for $R = 20$, $T = 6$ without (*upper curve*) and with (*lower*) a random disturbance of magnitude 0.01.

VIII. CONCLUDING REMARKS

The foregoing calculations have had two purposes. The first was to bring out the physical mechanism by which thermal dissipation destabilizes oscillations; this part of the problem is relatively simple and needs no further discussion. The second purpose was to explore the non-linear behavior of an oscillator which, like pulsationally unstable stars, depends on thermal dissipation for maintaining its amplitude of oscillation, and whose governing equation is therefore third order in time. We should like to conclude this work with some comments on the possible mathematical analogies.

The oscillatory motion of a star is characterized by three time scales: a dynamical (or free-fall) time, a signal propagation time (which measures the speed of pressure adjustments), and a radiative relaxation time. Similarly, as discussed at the beginning of § VI, the overstable oscillator is described by three analogous times: the free-fall time, the period of the spring, and the radiative time. Thus, we can imagine that the nature of stellar oscillations should be roughly predictable according to what region of the (R,T)-plane the star lies in, where R and T, as above, measure ratios of the various times.

The case of the star is somewhat simplified by the restriction that even pulsating stars seem always to be in approximate hydrostatic equilibrium. It is easy to show that this implies that the dynamical and pressure-adjustment times must be comparable. Thus in the (R,T)-plane, stars lie roughly in the region $R \sim T$. Even with this restriction, a wide range of phenomena are possible on the basis of one instability mechanism.

The analogy we are making here is just a mathematical one, and it is meant merely to indicate the kinds of phenomena that may be expected on the basis of one simple instability mechanism. Our results show that, if we treat simple instability mechanisms in detail, we can expect surprising range of phenomena. Many of these phenomena are similar to those observed in stars, and the work outlined here suggests these might occur without the aid of elaborate physical mechanisms. For example, it has been suggested in the past that recurrent novae exhibit relaxation oscillations. Here, we see that, from the same instability that provides regular oscillations, we get not only relaxation oscillations but also the superposed transient oscillations (squegging) which are so characteristic of novae. The same instability leads to irregular variations and a variety of asymmetries, and suggests a natural explanation of irregular variations of stars. (See the discussion at the end of § VII.) We are therefore led to conclude that a great number of phenomena observed in variable stars can be explained by the instability mechanism discussed by Eddington, once non-linear, non-adiabatic solutions can be found. We feel, however, that progress in this direction can be made through the study of elementary prototype equations perhaps more closely related to the stellar model than ours, but hopefully not more complicated. (Some of these already exist as discussed by Ledoux and Walraven [1958], and a very interesting prototype stability equation has been discussed by Baker [1965].)

Finally we should like to suggest that convective overstability itself, which is fairly well represented by our model, may play a role in certain types of variable stars. For example, the instability of the acoustic (p-) modes seems to provide a qualitative explanation of the semiregular variables. For these stars R is known to be quite large on account of the thickness of their convection zones, and T is reasonably large as well, by arguments of hydrostatic equilibrium. It may well be that the semiregular variables are acoustically overstable, in the sense discussed above, and lie moreover in the aperiodic region of the R-T-plane. This would account both for their variability and for their irregularity, but further calculations will be required to make this suggestion more precise.

We would like to thank Mrs. O. Ko for her skilful programming and Dr. N. H. Baker and Professor P. Ledoux for their interest in the work and for some stimulating discussions.

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